p-groups with some regularity properties

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p-Groups with some regularity properties (*)

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ABSTRACT. – We show that several properties of the regular p-groups hold for the p-groups in which the center contains all the elements of order p, if p is odd, all the elements of order 4, if $p = 2$ (TH-p-groups). We give examples of non regular TH-p-groups and characterize the p-groups G, with p odd, in which $|\, \Omega_1(G) \,| = p^2$.

KEY WORDS: *p-groups, Regular p-groups, Metacyclic p-groups.*

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1. Introduction

Some nice properties concerning the power structure of the regular p-groups hold also in some classes of non regular p-groups. We focus our attention on the p-groups in which all the elements of order p are central. Any p-group G, with p odd, satisfying such a condition is said here to be a *TH-p-group* in acknowledgment to Thompson. Several papers deal with such groups ([1], [3], [6], [7], [9]), and many results can be extended to the even case if the elements of order 2 and 4 are central. So, if $p = 2$, we call a group in which all the elements of order 2 and 4 are central a *TH-p-group*. Under this definition the results about TH-p-groups with odd p can be extended to the even case.

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In this paper we consider only *finite p-groups*. We use standard notation. In particular, if G is a p-group, we shall denote by $\Omega_i(G)$ the subgroup of G generated by its elements of order dividing p^i; by $\mho_i(G)$ the subgroup of G generated by the elements x^{p^i} with x in G; by $Z_i(G)$ and $\gamma_i(G)$ respectively the terms of upper and lower central series of G; we also denote by $\Phi(G)$ the Frattini subgroup of G and by $d(G)$ the minimal number of generators of G. For simplicity, when it will not be misleading, we shall write Ω_i, $\mho_i$, Z_i and γ_i instead of $\Omega_i(G)$, $\mho_i(G)$, $Z_i(G)$ and $\gamma_i(G)$.

One of Thompson's classical results says:

1.1. – *If G is a finite TH-p-group, with $p > 2$, then $d(G) \leq d(Z(G))$.*

This result has been extended to the case $p = 2$ by Mann (Theorem 1 in [8]). An argument in Blackburn's proof of this theorem (see [4] III, 12.2) and in Mann's proof of Theorem 1 in [8], for p=2, also shows that:

1.2. – *If G is a finite TH-p-group, then G/Ω_1 is a TH-p-group.*

This result has been generalized by Buckley [3] in the following theorem.

1.3. – *Let G be a TH-p-group of exponent p^n with $p > 2$. Let $1 \leq k \leq n$. Then*

(i) *G/Ω_k is a TH-p-group of exponent p^{n-k};*

(ii) $\Omega_k = \{x \mid x^{p^k} = 1\}$;

(iii) *The series $1 \leq \Omega_1 \leq \Omega_2 \leq \ldots \leq \Omega_n = G$ is central and hence the nilpotency class of G is at most n;*

(iv) $(xy)^{p^{n-1}} = x^{p^{n-1}} y^{p^{n-1}}$ *for all x, y in G.*

Laffey in [6] reaches similar results by an independent argument. He sets $\Omega(G) = \begin{cases} \Omega_1(G) & \text{if } p > 2 \\ \Omega_2(G) & \text{if } p = 2 \end{cases}$ and proves

1.4. – *Let G be a p-group with $\Omega(G') \leq Z(G)$. Then G/L is a TH-p-group for all subgroups L with $\Omega(G') \leq L \leq Z(G)$.*

As an easy consequence of this lemma, Laffey shows in particular that Proposition 1.3 (ii) *holds in the case $p = 2$,* also. The cited result of Mann allows us to extend the other items of 1.3 to the case $p = 2$ (see 2.1, 2.3).

In this paper, we consider the relations between the class of TH-p-groups and the class of regular p-groups. We prove that several properties of regular p-groups hold for the TH-p-groups (see 2.3, 2.7, 2.8). The class of regular TH-p-groups is quite large. For example, all TH-p-groups of exponent less than p^p are regular (see 2.4 (ii)). Also, all TH-p-groups in which $|\Omega_1| \leq p^p$ are absolutely regular (see 2.9). However there are examples of non regular TH-p-groups (see 4.1 and 4.2: the Example 4.2 was suggested by A. Caranti). In Section 3 we characterize the TH-p-groups with $|\Omega_1| = p^2$. This allows us, for $p > 2$ to obtain a characterization of the p-groups with $|\Omega_1| = p^2$.

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2. TH-p-groups and regular p-groups

We call a p-group G with $\Omega(G) \leq Z(G)$, where

$$\Omega(G) = \begin{cases} \Omega_1(G) & \text{if } p > 2 \\ \Omega_2(G) & \text{if } p = 2\,, \end{cases}$$

a *TH-p-group*. Obviously G is a TH-p-group if and only if $\Omega(G) = \Omega(Z(G))$ and, by Buckley's and Laffey's results cited above, all the elements of $\Omega_i(G)$ in a TH-p-group G have orders at most p^i. It is clear that *every subgroup of a TH-p-group is a TH-p-group* and that *the direct product of TH-p-groups is a TH-p-group.*

We observe that, according to our definition, the quaternion group and the generalized quaternion groups are not TH-p-groups. For $n \leq 3$ there are no non abelian TH-p-groups of order p^n; the only non abelian TH-p-group of order p^4 is $G = \langle a, b \mid a^{p^2} = b^{p^2} = 1,\, b^{-1}ab = a^{1+p} \rangle$. Since the quotient of such a group G with respect to its subgroup $N = \langle b^p \rangle$ is non abelian of order p^3, *the class of TH-p-groups is not closed under quotients.* On the other hand, it turns out that some remarkable quotients of TH-p-groups are themselves TH-p-groups (see, for example, 1.3 (i) and 1.4). In particular, from 1.4, using induction on k, it follows easily that *if G is a TH-p-group, then G/Z_k is a TH-p-group for every k.* Moreover, as a consequence of 1.3 (ii), which has been proved by Laffey for every prime p, we can restate 1.3 (i) for all p.

THEOREM 2.1. – *Let G be a TH-p-group of exponent p^n. Let $1 \leq k \leq n$. Then G/Ω_k is a TH-p-group of exponent p^{n-k}.*

Proof. – We proceed by induction on k. The first step is 1.2. We can assume that G/Ω_{k-1} is a TH-p-group of exponent p^{n-k+1}. By 1.2 we have that $\frac{G/\Omega_{k-1}}{\Omega_1(G/\Omega_{k-1})}$ is a TH-p-group of exponent p^{n-k}. Now, since the elements of Ω_k are all of order at most p^k, we have $\Omega_1(G/\Omega_{k-1}) = \frac{\Omega_k}{\Omega_{k-1}}$. Thus $G/\Omega_k \cong \frac{G/\Omega_{k-1}}{\Omega_k/\Omega_{k-1}} = \frac{G/\Omega_{k-1}}{\Omega_1(G/\Omega_{k-1})}$ is a TH-p-group of exponent p^{n-k}.

LEMMA 2.2. – *Let G be a TH-p-group. Then, for all i and k,*

$$\Omega_i(G/\Omega_k) = \frac{\Omega_{i+k}}{\Omega_k}.$$

Proof. – Let $x\Omega_k \in \Omega_i(\frac{G}{\Omega_k})$. Since G/Ω_k is a TH-p-group, by 1.3 (ii), $(x\Omega_k)^{p^i} = \Omega_k$. This implies $x^{p^i} \in \Omega_k$ and hence $x \in \Omega_{i+k}$. Conversely, if $x\Omega_k \in \frac{\Omega_{i+k}}{\Omega k}$, we have $x \in \Omega_{i+k}$ and then 1.3 (ii) implies $x^{p^{i+k}} = 1$. Thus $x^{p^i} \in \Omega_k$ and $x\Omega_k \in \Omega_i(\frac{G}{\Omega_k})$.

THEOREM 2.3. – *Let G be a TH-p-group. Then the series*

$$1 \leq \Omega_1 \leq \Omega_2 \leq \ldots \leq \Omega_n = G$$

is central with elementary abelian factors of non increasing orders (see [4] III, 10.7 b)).

Proof. – From 2.1 and 2.2 follows

$$\frac{\Omega_i}{\Omega_{i-1}} = \Omega_1(G/\Omega_{i-1}) \leq Z(G/\Omega_{i-1}).$$

Thus the series is central and, by 1.3 (ii), the factors $\frac{\Omega_i}{\Omega_{i-1}}$ are elementary abelian. Since Ω_2 is a TH-p-group, by 1.1 we have $d(\Omega_2(G)) \leq d(\Omega_1(\Omega_2(G))) = d(\Omega_1(G))$, hence

$$d\left(\frac{\Omega_2(G)}{\Omega_1(G)}\right) \leq d(\Omega_1(G)). \tag{1}$$

Now, by 2.2, $\frac{\Omega_{i+1}}{\Omega_i} \cong \frac{\Omega_{i+1}/\Omega_{i-1}}{\Omega_i/\Omega_{i-1}} = \frac{\Omega_2(G/\Omega_{i-1})}{\Omega_1(G/\Omega_{i-1})}$ holds for all i. Thus, by (1) with G/Ω_{i-1} instead of G, it turns out that

$$d\left(\frac{\Omega_{i+1}}{\Omega_i}\right) \leq d(\Omega_1(G/\Omega_{i-1})) = d\left(\frac{\Omega_i}{\Omega_{i-1}}\right).$$

As an immediate consequence of Theorem 2.3 we have the following

COROLLARY 2.4. – *Let G be a TH-p-group of exponent p^n. Then*

(i) *the nilpotency class of G is at most n. In particular, if the exponent of G is at most p^3, then G is metabelian;*
(ii) *if $n \leq p$, then G is regular;*
(iii) *$[\gamma_k, \Omega_i] \leq \Omega_{i-k}$ for all k and all $i \leq n$, provided $\Omega_j = 1$ whenever $j \leq 0$.*

COROLLARY 2.5. – *Let G be a TH-p-group of exponent p^n. Then*

(a) *$C_G(x) \geq \gamma_i(G)$ for all i and all $x \in \Omega_i$;*
(b) *$G^{(k)} \leq \Omega_{n+1-2^k}$ for all k.*

Proof. – (a) $[\gamma_i, \Omega_i] \leq \Omega_0 = 1$ follows from 2.4 (iii).

(b) By induction on k. If $k = 1$, $G' = [\gamma_1, \Omega_n] \leq \Omega_{n-1}$ follows directly from 2.4 (iii). Since $G^{(k)} \leq \gamma_{2^k}$ holds in every group G, the inductive hypothesis allows us to write

$$G^{(k+1)} = [G^{(k)}, G^{(k)}] \leq [\gamma_{2^k}, \Omega_{n+1-2^k}] \leq \Omega_{n+1-2^{k+1}} .$$

As an immediate consequence of 2.5 (a) we also obtain

COROLLARY 2.6. – *The commutator subgroup of a TH-p-group is contained in the centralizer of any element of order at most p^2. In particular every TH-p-group with commutator subgroup of exponent at most p^2 is metabelian.*

In [1] the TH-p-groups, with odd p, are characterized as p-groups in which $x^p = y^p$ implies $xy = yx$. The same argument shows that in any TH-2-group $x^2 = y^2$ implies $xy = yx$. With the aid of this result we can estabilish the following properties of the TH-p-groups.

THEOREM 2.7. – *Let G be a TH-p-group. The the following properties hold for all s, k, n and for all the elements $x, y \in G$.*

$$\begin{array}{lllll}
(a) & x^{p^s} = y^{p^s} & \iff & x\Omega_s = y\Omega_s . \\
(b) & x^{p^s} = y^{p^s} & \iff & (xy^{-1})^{p^s} = 1 . \\
(c) & [x^{p^k}, y^{p^n}] = 1 & \iff & [x, y]^{p^{k+n}} = 1 .
\end{array}$$

Proof. – (a) By induction on s. In case $s = 1$, by the above observation, $x^p = y^p$ implies $xy = yx$. It follows that $(y^{-1}x)^p = (y^p)^{-1}x^p = 1$, thus $y^{-1}x \in \Omega_1$. Conversely $x = yz$, with $z \in \Omega_1 \leq Z(G)$, implies $x^p = (yz)^p = y^p z^p = y^p$.

Assume next that $x^{p^s} = y^{p^s}$, so that $(x^p)^{p^{s-1}} = (y^p)^{p^{s-1}}$ and, by the inductive hypothesis, $x^p\Omega_{s-1} = y^p\Omega_{s-1}$. So $\bar{x}^p = \bar{y}^p$, where $\bar{g}$ denotes the coset $g\Omega_{s-1}$, for all $g \in G$. Now, by the result in the case $s = 1$, the relation $\bar{x}^p = \bar{y}^p$ is equivalent to $\bar{x}\Omega_1(\bar{G}) = \bar{y}\Omega_1(\bar{G})$ in the TH-p-group $\bar{G} = G/\Omega_{s-1}$ and by 2.2 this in turn is equivalent to $x\Omega_s = y\Omega_s$.

(b) This is an immediate consequence of 1.3 (ii) and (a).

(c) The proof depends upon (b) and upon the fact that G/Ω_k is a TH-p-group and proceeds as in [4] III, 10.6 b).

Corollary 2.8. – *Let G be a TH-p-group. Then $|\frac{G}{\mho_s(G)}| \leq |\Omega_s|$ for all s.*

Proof. – The relation (a) of 2.7 allows us to define a bijection π_s : $x\Omega_s \mapsto x^{p^s}$ between G/Ω_s and the set of p^s-th powers of elements of G (such a bijection was introduced by Xu [9]). Hence $|\frac{G}{\Omega_s}| \leq |\mho_s|$.

This corollary also yields an alternative proof of the well known results of Thompson and Mann (see 1.1). In fact, if G is a TH-p-group, we have

$$|\Omega_1| \geq |G : \mho_1| \geq |G : \Phi(G)| = p^{d(G)} .$$

We recall that any group G is said to be *absolutely regular* whenever $[G : \mho_1] < p^p$ and that *every absolutely regular group is regular* (see [2]). Hence another immediate consequence of 2.8 is the following:

Corollary 2.9. – *All TH-p-groups with $|\Omega_1| < p^p$ are absolutely regular.*

3. p-groups G with $|\Omega_1(G)| = p^2$

In the last two sections we shall use the following arithmetical lemma.

Lemma 3.1. – *Let z, t, n, k be natural numbers and p a prime with z, p coprime and t, p coprime. Then there are two natural numbers v, w, coprime to p, such that:*

(i) $(1 + zp^n)^{p^k} = 1 + vp^{n+k}$ *except if $p = 2$ and $n = 1$;*
(ii) $(1 + zp^n)^t = 1 + wp^n$.

Proof. – (i) By induction on k. If $k = 1$ we have

$$(1+p^n z)^p = 1 + p^{n+1} z + \sum_{i=2}^{p-1} \binom{p}{i} p^{ni} z^i + p^{np} z^p .$$

It is clear that, except if $p = 2$ and $n = 1$, p^{n+2} divides all the terms in the sum after the second. In fact, if $p > 2$, then p divides $\binom{p}{i}$ for all i with $2 \leq i \leq p-1$, moreover, since $n \geq 1$, we have $in \geq 2n \geq n+1$; on the other side $p \geq 3$ yields $pn \geq 3n \geq n+2$. Similarly, in the case $p = 2$ and $n \geq 2$ we have $2n \geq n+2$. Thus there is a natural number l such that

$$(1+p^n z)^p = 1 + p^{n+1} z + p^{n+2} l = 1 + p^{n+1}(z + pl)$$

and the proposition is proved with $v = z + pl$.

Assume next that the lemma is true for $k-1$, then we can suppose that there is a natural number $\bar{v}$, coprime to p, such that

$$(1+p^n z)^{p^k} = ((1+p^n z)^{p^{k-1}})^p = (1+p^{n+k-1}\bar{v})^p .$$

However, by using the case $k = 1$, there is a natural number v, coprime to p, such that

$$(1+p^{n+k-1}\bar{v})^p = 1 + p^{n+k} v$$

and hence the lemma holds for k.

(ii) We have

$$(1+zp^n)^t = 1 + p^n tz + \sum_{i=2}^{t} \binom{t}{i} p^{ni} z^i$$

and p^{n+1} divides all the terms in the sum after the second, since $ni \geq 2n \geq n+1$ for all $i \geq 2$. Thus there is a natural number l such that

$$(1+zp^n)^t = 1 + p^n tz + p^{n+1} l = 1 + p^n (tz + pl)$$

and, as tz is coprime to p, then so is $w = tz + pl$.

LEMMA 3.2. – *All the TH-2-groups with $|\Omega_1| = 2^2$ are metacyclic.*

Proof. – Let G be a minimal counterexample. Then, G is not abelian and, by [4] III,11.11, $|G| \leq 2^5$. It follows that $|Z(G)| \leq 2^3$ and, since $\Omega_1 < \Omega_2 \leq Z(G)$, we have $\Omega_2 = Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Hence G/Z is elementary abelian of order 4 and, by 2.2, we have

$$G/\Omega_2 = \Omega_1(G/\Omega_2) = \Omega_3/\Omega_2 .$$

Thus $G = \Omega_3$. But the factors in the Ω_i-series are of non increasing orders, so

$$|\Omega_3/\Omega_2| \leq |\Omega_2/\Omega_1| = 2$$

which contradicts $|G/Z| = 4$.

THEOREM 3.3. – *G is a TH-p-group with $|\Omega_1(G)| = p^2$ if and only if it is metacyclic and it has the following presentation:*

$$G = \langle a, b \mid a^{p^m} = 1,\ b^{p^n} = a^{p^{m-s}},\ a^b = a^{1+p^{m-r}} \rangle$$

where the parameters satisfy one of the following system of conditions:

(i) $0 = s \leq r < \min\{n, m\}$,
(ii) $p = 2,\ m \geq 2,\ n \geq 2,\ 0 = s \leq r < \min\{n-1, m-1\}$,
(iii) $\max\{1, m-n+1\} \leq s < \min\{r, m-r\}$,
(iv) $p = 2,\ \max\{1, m-n+1\} \leq s < \min\{r, m-r-1\}$.

(*The conditions* (i) *and* (ii) *concern splitting groups, the others non splitting groups in the sense of* [5].)

Proof. – Let G be a TH-p-group, with $|\Omega_1(G)| = p^2$. Then G is not cyclic and, by 2.8, $|G/\mho_1| \leq p^2$. Hence, if $p > 2$, G is metacyclic (see [4] III, 11.4); if $p = 2$, we obtain the same conclusion by Lemma 3.2. Thus, by [5], G has a presentation of the form

$$G = \left\langle a, b \mid a^{p^m} = 1,\ b^{p^n} = a^{p^{m-s}},\ a^b = \begin{cases} a^{1+p^{m-r}} \\ a^{-1+p^{m-r}} \end{cases} \right\rangle \tag{2}$$

under certain conditions on the parameters. The case $a^b = a^{-1+p^{m-r}}$ occurs only when $p = 2$, $r < m-1$ and it is said to be *exceptional* by King.

In the exceptional case it turns out that $a^{2^{m-2}} \notin Z(G)$, in fact $(a^{2^{m-2}})^b = a^{2^{m-2}}$ is equivalent to

$$(-1 + 2^{m-r})2^{m-2} \equiv 2^{m-2} (\bmod\ 2^m);$$

since $m - r > 1$, this congruence has no solution. Thus there is no TH-2-group among exceptional metacyclic 2-groups.

In ordinary metacyclic p-groups the parameters which appear in (2) are linked by the conditions

$$0 = s \leq r < \min\{n+1, m\} \quad \text{splitting case}, \tag{3}$$

$$\max\{1, m-n+1\} \leq s < \min\{r, m-r+1\} \quad \text{non splitting case} \tag{4}$$

(see [5]); moreover in both cases, if $p = 2$ and $m \geq 2$, we must have $r < m-1$.

We shall see that $Z(G) = \langle a^{p^r}, b^{p^r} \rangle$ (see [5] 4.10 (ii)). First of all we observe that an element $b^x a^y \in G$ belongs to $Z(G)$ if and only if b^x and a^y are in $Z(G)$. Now a^{p^h}, b^{p^k} are respectively the smallest powers of a and b belonging to $Z(G)$ if and only if h and k are the least positive integers with $(a^{p^h})^b = a^{p^h}$ and $(a^{b^k}) = a$. So h is a solution of the congruence

$$(1 + p^{m-r})p^h \equiv p^h (\bmod\ p^m)$$

and hence $h = r$. Similarly k is a solution of the congruence

$$(1 + p^{m-r})^{p^k} \equiv 1 (\bmod\ p^m) .$$

By Lemma 3.1 (note that we have $m - r > 1$ in case $p = 2$), we obtain $(1 + p^{m-r})^{p^k} = 1 + vp^{m+k-r}$ with $(v, p) = 1$. Hence it follows from (5) that $m \leq m + k - r$ and so $k = r$.

By (3) and (4) we see that $a^{p^{m-1}} \in Z(G)$; moreover, since G is a TH-p-group, there exists an element $c = b^{xp^r} a^{yp^r} \in Z(G)$ of order p such that $\langle c \rangle \cap \langle a \rangle = 1$. This last condition implies

$$xp^r \not\equiv 0 (\bmod\ p^n) , \tag{6}$$

and hence $r < n$. Thus, in the splitting case, by (3), (i) holds.

Next, from $1 = c^p = b^{xp^{r+1}} a^{yp^{r+1}}$ it follows that

$$b^{xp^{r+1}} = a^{-yp^{r+1}} \in \langle a \rangle \cap \langle b \rangle = \langle b^{p^n} \rangle .$$

Hence $xp^{r+1} \equiv 0 (\bmod\ p^n)$. In view of (6), we can write $x = tp^{n-r-1}$ with $(p, t) = 1$, and $c^p = 1$ implies $a^{tp^{m-s} + yp^{r+1}} = 1$, that is

$$tp^{m-s} + yp^{r+1} \equiv 0 \ (\bmod\ p^m) .$$

In this congruence there is a solution for y if and only if p^{r+1} divides tp^{m-s} and, since $(t, p) = 1$, this implies $r + 1 \leq m - s$. In the non splitting case, by (4), this in turn implies (iii).

Finally let $p = 2$. In the splitting case, if $m \geq 2$, we can also assume $n \geq 2$. Indeed, if $n = 1$, by (i), we must have $r = 0$, and these groups are isomorphic to the ones with parameters $r = 0$, $m = 1$, $n \geq 2$. Since $a^{2^{m-2}} \in Z(G)$ and $b^{2^{n-2}} \in Z(G)$, (ii) follows. In the non splitting case, (iii) implies $m \geq 2$ and $n \geq 2$. Moreover the element $c_1 = b^{2^{n-2}} a^{-2^{m-s-2}}$ is of order 4, since, by (iii), $b^{2^{n-2}}$ is central. We argue that also $a^{-2^{m-s-2}}$ is central; hence $r \leq m - s - 2$ and (iv) holds.

Conversely suppose G is metacyclic with parameters satisfying conditions (i)-(iv). We have to prove that $\Omega(G) \leq Z(G)$ and that $|\Omega_1(G)| = p^2$.

We consider first the case $p > 2$ and we prove that $|\Omega_1(G) \cap Z(G)| \geq p^2$. Obviously $a^{p^{m-1}} \in \Omega_1(G) \cap Z(G)$ and, since $r < n$, $r < m - s$, we also have $c = b^{p^{n-1}} a^{-p^{m-s-1}} \in \Omega_1(G) \cap Z(G)$. Next we have $\langle a \rangle \cap \langle c \rangle = 1$, otherwise $c \in \langle a \rangle$ and then $b^{p^{n-1}} \in \langle a \rangle$, which contradicts the meaning of the parameter n. Now it will suffice to show that $|\Omega_1(G)| = p^2$. Indeed, since G is regular (see [4] III, 10.2.c)) we have $|\Omega_1(G)| = |G/\mho_1(G)| = p^2$ (see [4] III, 10.7.a)).

If $p = 2$ and $m = 1$, G is the direct product of two cyclic groups and everything is clear.

In the case $p = 2$, $m \geq 2$ we shall prove that every element of order 4 is central. Let $c_1 = b^x a^y \in G$. We can write

$$\begin{aligned} c_1^2 &= (b^x a^y)^2 = b^{2x} (a^y)^{b^x} a^y = b^{2x} a^{((1+2^{m-r})^x + 1)y} \\ &= b^{2x} a^{\left(2 + \sum_{i=1}^{x} \binom{x}{i} 2^{i(m-r)}\right) y} = b^{2x} a^{2ty}, \end{aligned}$$

where, since 4 divides $\sum_{i=1}^{x} \binom{x}{i} 2^{i(m-r)}$, $t = 1 + \frac{1}{2} \sum_{i=1}^{x} \binom{x}{i} 2^{i(m-r)}$ is odd. Similarly we have

$$c_1{}^4 = (b^{2x} a^{2ty})^2 = b^{4x} a^{4t_1 y}$$

with odd t_1. So $c_1{}^4 = 1$ implies $4x \equiv 0 \pmod{2^n}$ and, if we set $x = h2^{n-2}$, we also have

$$h2^{m-s} + 4yt_1 \equiv 0 \pmod{2^m}.$$

It follows that $y \equiv 0 \pmod{2^{m-s-2}}$ and, since $r \leq n-2$ and $r \leq m-s-2$ (see (ii) and (iv)), we conclude that c_1 is central. As G is not cyclic and $Z(G) = \langle a^{2^r}, b^{2^r} \rangle$, we can now argue that $|\Omega_1(G)| = 2^2$.

Corollary 3.4. – *The following three conditions on a p-group G, with $p > 3$, are equivalent:*

(a) *G is a non-split extension of a metacyclic TH-p-group by a cyclic group of order at most p.*
(b) $|\Omega_1(G)| = p^2$.
(c) *G is metacyclic non cyclic.*

Proof. – (a) $\Rightarrow$ (b) Let G be a non-split extension of a metacyclic TH-p-group C by a cyclic group of order at most p. Every element of order p belongs to C, otherwise G would split on C. Hence $\Omega_1(G) = \Omega_1(C)$ and then $|\Omega_1(G)| = p^2$ by 3.3.

(b) $\Rightarrow$ (c) Let $|\Omega_1(G)| = p^2$, then G contains $p+1$ subgroups of order p. Since $p > 3$, by [4] III, 11.6, G is metacyclic.

(c) $\Rightarrow$ (a) If G is metacyclic, then, by 3.3, $|\Omega_1(G)| = p^2$. Hence G contains an abelian normal subgroup of type (p, p) but has no abelian subgroups of type (p, p, p). Set $C = C_G(\Omega_1(G))$. By [10] 4.12, C is metacyclic and $|G : C| \leq p$. Since $\Omega_1(C) = \Omega_1(G)$, it turns out that C is a TH-p-group and there is no element $x \in G$ of order p with $G = \langle x, C\rangle$. This means that G cannot split on C.

REMARK 3.5. – The conditions (a) and (b) of the previous corollary are still equivalent, for $p = 3$.

On the other hand, there exist non metacyclic p-groups G with $p = 2, 3$ such that $|\Omega_1(G)| = p^2$. For $p = 3$ see for example the well known construction of an irregular p-group of order p^{p+1} given by Blackburn ([4] III, 10.15).

Note that, actually, the non metacyclic 3-group G with $|\Omega_1(G)| = 3^2$ are the only non regular ones (see [4] III, 10.2 c) and 11.4).

For $p = 2$ we can consider $G = \frac{D_4 \times \langle c\rangle}{H}$ where

$$D_4 = \langle a, b \mid a^4 = b^2 = 1,\ [a, b] = a^2\rangle\,,$$

$\langle c\rangle$ is cyclic of order 4 and $H = \langle a^2c^{-2}\rangle$. Then $|\Omega_1(G)| = 2^2$, but, since $|G/\Phi(G)| = 2^3$, G is non metacyclic.

4. Non regular TH-p-groups

The following class of non regular TH-3-groups is obtained adapting a well known construction by Wielandt (see [4] III, 10.3 c)).

4.1. – *Let $G_i = \langle a_i, b_i \mid {a_i}^{3^m} = 1,\ {b_i}^{3^n} = {a_i}^{3^{m-s}},\ {a_i}^{b_i} = {a_i}^{1+3^{m-r}}\rangle$ for $i = 1, 2$ be a metacyclic TH-3-group with $m < 2r$; then $G = G_1 \times G_2$ is a non regular TH-3-group.*

Proof. – We consider the subgroup $H = \langle h_1, h_2\rangle \leq G$ where $h_1 = a_1{b_2}^{-1}$ and $h_2 = {b_1}^{-1}a_2$. To establish that H is a non regular 3-group, it will suffice to show that its derived subgroup is not cyclic (see [4] III, 10.3 b)). Since $b_ia_i{b_i}^{-1} = {a_i}^{(1+3^{m-r})^{(3^{n+s}-1)}}$, by 3.1 (ii) we get $(1 + 3^{m-r})^{(3^{n+s}-1)} = 1 + w3^{m-r}$, where $(w, 3) = 1$. It follows that

$$(7) \qquad b_ia_i{b_i}^{-1} = {a_i}^{1+w3^{m-r}}, \quad \text{with} \quad (w, 3) = 1\,.$$

Let $c = [h_1, h_2]$; then $c = [a_1, b_1{}^{-1}][b_2{}^{-1}, a_2] = a_1{}^{-1} a_1{}^{b_1{}^{-1}} a_2{}^{-b_2{}^{-1}} a_2$ and by (8) we obtain $c = a_1{}^{w3^{m-r}} a_2{}^{-w3^{m-r}}$. It turns out that

$$[c, h_1] = [a_1{}^{w3^{m-r}} a_2{}^{-w3^{m-r}}, a_1 b_2{}^{-1}] = [a_2{}^{-w3^{m-r}}, b_2{}^{-1}]$$
$$= a_2{}^{w3^{m-r}} (a_2{}^{1+w3^{m-r}})^{-w3^{m-r}} = a_2{}^{-w^2 3^{2(m-r)}} .$$

Now, since $m < 2r$, we observe that c and $[c, h_1]$ cannot be one a power of the other.

EXAMPLE 4.2. – Let $p > 2$ be a prime, s a natural number with $s > p$. We consider the group $A = \langle a_1 \rangle \times \cdots \times \langle a_p \rangle$, where each $\langle a_i \rangle$ is cyclic of order p^s. Define $\alpha \in \operatorname{Aut} A$ by

$$a_1^\alpha = a_1 ,$$
$$a_i^\alpha = a_{i-1} a_i \quad \text{for} \quad i = 2, \ldots , p .$$

Since α induces a p-automorphism on $A/\Phi(A)$, it follows that α is a p-automorphism. Let p^t be the order of α and $\langle y \rangle$ a cyclic group of order p^{t+1}. We define the action of $\langle y \rangle$ on A by $a_i{}^y = a_i^\alpha$ for $i = 1, \ldots , p$. The subgroup $K = \langle a_1{}^{p^{s-p+1}}, a_2{}^{p^{s-p+2}}, \ldots , a_{p-1}{}^{p^{s-1}} \rangle$ is normalized by $\langle y \rangle$, since for each $i = 1, \ldots , p-1$

$$(a_i{}^{p^{s-p+i}})^y = a_{i-1}{}^{p^{s-p+i}} a_i{}^{p^{s-p+i}} = (a_{i-1}{}^{p^{s-p+i-1}})^p a_i{}^{p^{s-p+i}} \in K .$$

Thus $\langle y \rangle$ acts on A/K. We will show that the semidirect product $G = [A/K]\langle y \rangle$ is a non regular TH-p-group. First of all we show that $\Omega_1(G) = \Omega_1(A/K)\Omega_1\langle y \rangle$. If $y^k x \in G$ with $x \in A/K$, then we have $(y^k x)^p = y^{kp} x^p u$ for a suitable $u \in [A/K, \langle y \rangle] \leq A/K$; then $(y^k x)^p = 1$ gives $y^{kp} = 1$, that is $y^k \in \langle y^{p^t} \rangle$. But $a_i{}^{y^{p^t}} = a_i{}^{\alpha^{p^t}} = a_i$, so $\langle y^{p^t} \rangle \leq Z(G)$. Hence, from $(y^k x)^p = y^{kp} x^p = 1$ we get $x \in \Omega_1(A/K)$.

Denoting $xK \in A/K$ simply by $\bar{x}$, we have

$$(\bar{a}_i{}^{p^{s-p+i-1}})^y = \bar{a}_{i-1}{}^{p^{s-p+i-1}} \bar{a}_i{}^{p^{s-p+i-1}} = \bar{a}_i{}^{p^{s-p+i-1}} .$$

Therefore $\Omega_1(A/K) \leq Z(G)$ and $\Omega_1(G) \leq Z(G)$.

We prove now that it is impossible to write $(y\bar{a}_p)^p$ as a product of p-th powers of elements of G. We have

$$(y\bar{a}_p)^p = y^p \bar{a}_p{}^{y^{p-1}} \bar{a}_p{}^{y^{p-2}} \cdots \bar{a}_p{}^y \bar{a}_p$$

where, for $n < p$,

$$\bar{a}_p{}^{y^n} = \bar{a}_{p-n} \bar{a}_{p-n+1}^{\binom{n}{n-1}} \bar{a}_{p-n+2}^{\binom{n}{n-1}} \cdots \bar{a}_{p-1}^{n} \, \bar{a}_p ;$$

in particular

$$\bar{a}_p{}^{y^{p-1}} = \bar{a}_1 \bar{a}_2{}^{\binom{p-1}{p-2}} \cdots$$

and $\bar{a}_1$ does not occur in $\bar{a}_p{}^{y^n}$ for $n \neq p-1$.

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